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Spectra of pinned charge density waves with background current

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Abstract

We develop techniques which allow us to calculate the spectra of pinned charge density waves with background current in one-dimensional space. We show that in such systems the low-lying modes are always localized, and we compute their spectral density and the localization length. We also show that as the energy is increased, the modes delocalize in a way similar to that in the Hatano–Nelson non-Hermitian quantum mechanics.

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1. Introduction

The problem of a quantum mechanical particle whose motion is governed by a random time-independent Hamiltonian has been discussed in the literature for several decades. Because of its relevance to the electrons' motion in disordered conductors, the Hamiltonian which consists of a kinetic energy and a random scalar potential is probably the most famous example of such a problem. It is now well established that the wavefunctions of such Hamiltonians whose energies lie within certain energy bands are localized, a phenomenon usually referred to as Anderson localization [1]. A wealth of new types of behaviour was discovered once random Hamiltonians constrained by certain symmetries were studied. For example, the most general random Hamiltonian would be complex Hermitian. Constraining it to be real (in other words, imposing time-reversal invariance) changes the localization properties of its wavefunctions. An electron moving in a disordered conductor is described by a real random Hamiltonian. Turning on a magnetic field makes the Hamiltonian Hermitian. This made the study of the crossover between real Hermitian and complex Hermitian random Hamiltonians easily accessible experimentally, and a subject of intense theoretical investigations [2].

Furthermore, it was discovered that a number of other symmetries can also be imposed on random Hamiltonians, which change their behaviour yet again. Altogether, there are ten symmetry classes of random Hamiltonians, distinguished by nine different constraints imposed on them [3, 4].

Yet other types of random Hamiltonians arise when they are constrained not by a symmetry but by certain requirements their spectra must satisfy. The most prominent example of that would be systems with bosonic excitations [5]. To be specific, consider an energy functional $E[\phi(x)]$, where $\phi(x)$ is some function, which has the form

$$E[\phi(x)] = \int_0^L \mathrm{d}x \left[\frac{1}{2} \left(\frac{\mathrm{d}\phi(x)}{\mathrm{d}x} \right)^2 + h(\phi(x), x) \right]. \tag{1}$$

Here $h(\phi(x), x)$ is a random function of its two arguments, one of them being the value of the function $\phi(x)$ at the point x, and the other being the point x itself. h can also be thought of simply as a function of two independent variables, and in what follows we will sometimes use the notation $h(\phi(x), x) = h(\varphi, x)|_{\varphi=\phi(x)}$. The precise form of the function $h(\varphi, x)$ will be discussed below.

Suppose $\phi_0(x)$ is a minimum (local or global) of the functional E. We would like to study the normal modes ψ_n of oscillations around that minimum. They are defined as the eigenfunctions of the matrix of second derivatives of the functional $E[\phi(x)]$ at its minimum $\phi(x) = \phi_0(x)$, or more precisely

$$\int \mathrm{d}y \left. \frac{\delta^2 E}{\delta\phi(x)\delta\phi(y)} \right|_{\phi(x)=\phi_0(x)} \psi_n(y) = \epsilon_n \psi(x).$$

Working out the functional derivatives results in the normal mode equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi_n(x) + \partial_\phi^2 h(\phi_0(x), x)\psi_n(x) = \epsilon_n \psi_n. \tag{2}$$

Here the notation $\partial_\phi^2 h(\phi_0(x), x)$ is used to denote $\partial^2 h(\varphi, x)/\partial\varphi^2|_{\varphi=\phi_0(x)}$. Equation (2) has to be supplemented by suitable boundary conditions. For definiteness we are going to take $\psi_n(0) = \psi_n(L)$ for some large L, although the results presented in this paper are largely insensitive to the choice of boundary conditions.

Equation (2) contains random terms, therefore it is hopeless to try to solve it exactly. However, in most applications we only need to calculate the average density of states and localization length of equation (2). The average density of states is defined by

$$\rho(\epsilon) = \frac{1}{N} \left\langle \sum_{n=1}^{N} \delta(\epsilon - \epsilon_n) \right\rangle,$$

where N is the total number of states and the angular brackets denote averaging over random h. The localization length ξ_n, in case ψ_n is localized, is defined via the asymptotic behaviour

$$\psi_n(x) \sim \exp\left(-\frac{|x - x_0|}{\xi_n} \right), \qquad |x - x_0| \gg \xi_n,$$

where x_0 is the localization centre which itself depends on n. ξ can be thought of as a function of ϵ as in $\xi(\epsilon) = \xi_n$ if $\epsilon = \epsilon_n$.

Equation (2) is a random Schrödinger-like equation, so it is tempting to conclude that its density of states and localization length is given by a simple Anderson localization type behaviour. Yet it is not equivalent to a particle moving in an arbitrary random potential. $\partial_\phi^2 h(\phi_0(x), x)$ is a random function of x, but it is not an arbitrary random function. It is clear, for example, that $\epsilon_n \geqslant 0$ for all n (just as the second derivative of a function at a point of its minimum is always non-negative). This tells us that $\partial_\phi^2 h(\phi_0(x), x)$, although random, has to be constrained in such a way that all its eigenvalues are positive or equal to zero.

In most applications of equation (1), $h(\varphi, x)$ is chosen to be a smooth function of its first argument φ and a rough function of its second argument x. For example,

$h(\varphi, x) = A(x)\cos(\varphi - \chi(x))$, where $A(x)$ and $\chi(x)$ are random functions of x uncorrelated at different x and $\chi(x)$ is uniformly distributed between 0 and π. This leads to the two-point correlation function

$$\langle h(\varphi, x)h(\varphi', x')\rangle = \alpha\cos(\varphi - \varphi')\delta(x - x').$$

Under this definition of $h(\varphi, x)$, the problem described by equations (1) and (2) has been extensively studied in the context of pinned charge density waves. We refer the readers who would like to know more about charge density waves to [6] while their pinning by disorder is discussed in the classic work [7]. Knowing the density of states and localization length of equation (2) allows, for example, to calculate the ac conductance of the charge density waves. This is why equation (2), which is sometimes referred to as Fukuyama–Lee equation in the charge density waves context, has been extensively studied in the literature. It was first deduced in [8, 9] that the density of states $\rho(\epsilon)$ of equation (2) is given by $\rho(\epsilon) = \epsilon^{\frac{3}{2}}$ if $\epsilon \ll \epsilon_c$ and if $\phi_0(x)$ is a global minimum of the energy functional, equation (1), where $\epsilon_c \sim \alpha^{2/3}$ is the crossover scale. If, on the other hand, $\phi_0(x)$ is a local minimum of equation (1), then $\rho(\epsilon) \sim \epsilon, \epsilon \ll \epsilon_c$.

A more detailed approach to the problem specified by equation (2) was developed in [10]. It was shown in that work that the potential $\partial_\phi^2 h(\phi_0(x), x)$ of the 'Schrödinger' equation (2) can always be represented as

$$\partial_\phi^2 h(\phi_0(x), x) = \frac{\mathrm{d}V(x)}{\mathrm{d}x} + V^2(x),$$

where $V(x)$ is some new random function. As a result, equation (2) is equivalent to

$$\mathcal{H}\begin{pmatrix}\psi_n(x)\\ \phi_n(x)\end{pmatrix} = \omega_n\begin{pmatrix}\psi_n(x)\\ \phi_n(x)\end{pmatrix}, \tag{3}$$

where

$$\mathcal{H} = \begin{pmatrix}0 & \frac{\mathrm{d}}{\mathrm{d}x} + V(x)\\ -\frac{\mathrm{d}}{\mathrm{d}x} + V(x) & 0\end{pmatrix}, \tag{4}$$

and $\omega_n^2 = \epsilon_n$. Now $\mathcal{H}$ is an example of a random Hamiltonian constrained by a symmetry and can be solved by techniques developed in that context. The symmetry of $\mathcal{H}$ is usually referred to as chiral symmetry. It is expressed by the relation $\sigma_3\mathcal{H}\sigma_3 = -\mathcal{H}$, which holds true for any random $V(x)$ [3]. Here, σ_3 is the usual Pauli matrix. The density of states and localization properties of chiral random Hamiltonians are strikingly different from those of Anderson Hamiltonians [11].

The full solution of equation (2), with the help of the mapping to equation (3), demonstrated that there indeed exists the crossover energy scale ϵ_c. At $\epsilon_n \ll \epsilon_c$ all the eigenfunctions $\psi_n(x)$ are localized with localization length $\xi_n \sim \epsilon_c^{-\frac{1}{2}}$, which is independent of n. At $\epsilon_n \gg \epsilon_c$, the eigenfunctions are still localized, but with the localization length which increases with ϵ_n as $\xi_n \sim \epsilon_n$. The density of states at $\epsilon_n \gg \epsilon_c$ is given by $\rho(\epsilon) \sim \epsilon^{-\frac{1}{2}}$. Finally, the density of states at $\epsilon_n \ll \epsilon_c$ is given by $\rho(\epsilon) \sim \epsilon^{\frac{3}{2}}$ if $\phi_0(x)$ is a global minimum of equation (1) and by $\rho(\epsilon) \sim \epsilon$ if $\phi_0(x)$ is a local minimum of equation (1).

In this paper, we would like to show that the same techniques which proved useful in solving equation (2) can also be used to solve another related problem, which we formulate below. Consider an equation

$$-j\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - \gamma\frac{\mathrm{d}\phi}{\mathrm{d}x} + \mathrm{d}_\phi h(\phi(x), x) = 0, \tag{5}$$

where j and γ are some parameters, and $\partial_\phi h(\phi(x), x) = \partial h(\varphi, x)/\partial\varphi|_{\varphi=\phi(x)}$. If $\gamma = 0$, then this equation is equivalent to the minimization condition of the energy equation (1). If, on the

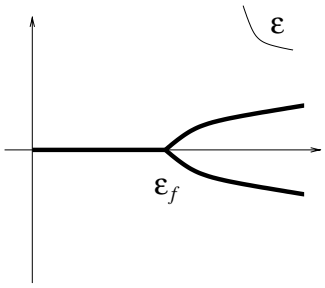

Figure 1. The eigenvalues ϵ_n of equation (6) lie on this curve in the complex plane.

other hand, $\gamma > 0$, then this defines a new problem. Equation (5) describes moving charge density waves, or charge density waves with background current [12–14]. Normal modes of oscillations of such pinned charge density wave are given by the equation

$$\left[-j\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \gamma\frac{\mathrm{d}}{\mathrm{d}x} + \partial_\phi^2 h(\phi_0(x), x)\right]\psi_n(x) = \epsilon_n\psi_n(x), \tag{6}$$

where $\phi_0(x)$ is a solution to equation (5), supplemented by the same boundary condition $\psi_n(0) = \psi_n(L)$. In this paper, we are going to present the solution to the problem defined by equations (5) and (6).

The operator in the square brackets of equation (6) is non-Hermitian. As a result, the eigenvalues ϵ_n do not have to be real. It is well known, however, that the eigenvalues of equations of the type equation (6) always lie along one-dimensional curves in the complex plane [15]. In this paper, we show that these curves take the shape depicted in figure 1. The fork point ϵ_f cannot be found exactly. However, it is still possible to define the crossover scale ϵ_c. All the states with energy less than ϵ_c have the same localization length ξ. As the energy is increased past ϵ_c, the localization starts to grow and eventually diverges at ϵ_f (which is always bigger than ϵ_c). The states corresponding to complex values of energy are delocalized.

We introduce the technique which allows us to compute ϵ_c and ξ. We calculate them in the regime where $\gamma/(j\alpha)^{1/3} \gg 1$ and find them to be $\epsilon_c = \gamma^2/4j$ and $\xi = 2\gamma^2/\alpha$. We also calculate the density of states along the part of the curve depicted in figure 1 which lies on the real axis and find it to be $\rho(\epsilon) \sim \epsilon$.

The rest of this paper is organized as follows. In section 2, we map the problem defined by equation (6) to a random chiral Hamiltonian. In section 3, the localization length of normal mode oscillations will be derived in the limit of large background current and in section 4 this result will be compared with the usual Larkin length of the problem. Section 5 contains the calculation of the density of the low-lying states. Finally, in section 6 we discuss the response of a pinned charge density wave with background current to an electric field using the formalism developed here.

2. Mapping to a chiral Hamiltonian

The first step of the solution is to map equation (6) to a more tractable form of a random chiral Hamiltonian. Following the general guidelines of [10], we treat equation (6) as the equation of motion of a particle with coordinate ϕ, moving in time x, in the presence of friction forces given by the γ term. We then pass from the Lagrangian to the Eulerian description of the motion. This involves introducing the velocity function $u(\phi(x)) \equiv \partial_x\phi(x)$. As is standard in hydrodynamics, u becomes a function of space ϕ and time x, regardless of the particular

particle trajectory $\phi(x)$. The equation of motion, equation (5), acquires the form familiar from fluid dynamics

$$j\partial_x u + ju\partial_\phi u + \gamma u = \partial_\phi h(\phi, x). \tag{7}$$

The equation thus obtained is similar to the equation of motion of a one-dimensional fluid without pressure, usually referred to as the Burgers equation. However, it also features the γu term, which can be interpreted as a kind of viscous friction. This friction term is different from a more standard Navier–Stokes viscosity term $-\partial^2 u/\partial\phi^2$.

Performing a derivative on equation (7) with respect to ϕ results in

$$j\partial_x\partial_\phi u + j(\partial_\phi u)^2 + ju\partial_\phi^2 u + \gamma\partial_\phi u = \partial_\phi^2 h. \tag{8}$$

We define the gradient of the velocity along the solution to equation (5), ϕ_0, as $F(x) \equiv \partial_\phi u(\phi_0(x), x)$. $F(x)$ can be related to $h(\phi, x)$ with the help of equation (8)

$$j\frac{\mathrm{d}F}{\mathrm{d}x} + jF^2 + \gamma F = \partial_\phi^2 h(\phi_0(x), x). \tag{9}$$

Thus equation (6) becomes

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\gamma}{j}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\mathrm{d}F}{\mathrm{d}x} + F^2 + \frac{\gamma}{j}F\right]\psi_n = \frac{\epsilon_n}{j}\psi_n, \tag{10}$$

or in a more symmetric form

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + F + \frac{\gamma}{j}\right]\left[-\frac{\mathrm{d}}{\mathrm{d}x} + F\right]\psi_n = \frac{\epsilon_n}{j}\psi_n. \tag{11}$$

The operator on the left-hand side of equation (11) is not Hermitian. This may be remedied by using a trick due to Hatano and Nelson [16]. Writing

$$\psi_n(x) = \exp\left(-\frac{\gamma}{2j}x\right)\tilde{\psi}_n(x) \tag{12}$$

the equation becomes

$$\left[\frac{\mathrm{d}}{\mathrm{d}x} + F + \frac{\gamma}{2j}\right]\left[-\frac{\mathrm{d}}{\mathrm{d}x} + F + \frac{\gamma}{2j}\right]\tilde{\psi}_n = \frac{\epsilon_n}{j}\tilde{\psi}_n. \tag{13}$$

Thus, we mapped our equation into an eigenvalue problem for a Hermitian operator. This operator can also be rewritten in the form similar to equation (3),

$$\mathcal{H}\begin{pmatrix}\tilde{\psi}_n(x)\\ \tilde{\phi}_n(x)\end{pmatrix} = \frac{\omega_n}{\sqrt{j}}\begin{pmatrix}\tilde{\psi}_n(x)\\ \tilde{\phi}_n(x)\end{pmatrix},$$

where $\mathcal{H}$ is still given by equation (4), $\omega_n^2 = \epsilon_n$, and

$$V(x) = F(x) + \frac{\gamma}{2j}.$$

However, as argued by Hatano and Nelson, the transformation equation (12) is valid only as long as $\tilde{\psi}_n$ decays asymptotically as $\exp(-|x|/\tilde{\xi})$, with the localization length $\tilde{\xi} < 2j/\gamma$. For $\tilde{\psi}_n$ whose localization length obeys this condition, equation (11) is equivalent to equation (13), and therefore, ϵ_n is real and positive. For other eigenfunctions $\tilde{\psi}_n$ whose localization length is larger than $2j/\gamma$, equation (13) is no longer equivalent to equation (11).

Note that the localization length $\tilde{\xi}_n$ of $\tilde{\psi}_n$ is related to the localization length ξ_n of ψ_n as in

$$\xi_n = \left(\frac{1}{\tilde{\xi}_n} - \frac{\gamma}{2j}\right)^{-1}, \tag{14}$$

as long as $\tilde{\xi} < 2j/\gamma$.

3. The localization length

Random chiral Schrödinger equations of the form (13) have been investigated by Comtet, Debois and Monthus [17]. It was determined that the important parameter for these equations is $\omega_c = \sqrt{j}\langle V(x)\rangle$, and correspondingly $\epsilon_c \equiv \omega_c^2$. For all wavefunctions whose energy $\omega_n < \omega_c$, the localization length is constant and is given by $\tilde{\xi} = 1/\langle V(x)\rangle$. For $\omega_n > \omega_c$, the localization length quickly increases, being asymptotically proportional to ω_n^2.

The crucial test for our theory is, therefore, whether $\langle V(x)\rangle$ is bigger or smaller than $\gamma/2j$. As we will see below, $\langle V(x)\rangle > \gamma/2j$. As a result equation (13) is equivalent to equation (11) and consequently to equation (6), at $\epsilon < \epsilon_c$. Thus, at $\epsilon < \epsilon_c$ all the eigenvalues of equation (6) are real and positive. On the other hand, there exist some $\epsilon_f > \epsilon_c$ where the localization length of $\tilde{\psi}_n$ becomes equal to $2j/\gamma$. At that point, the wavefunctions ψ_n become delocalized, in accordance with equation (14). At $\mathrm{Re}\,\epsilon > \epsilon_f$, the eigenvalues of equation (6) are no longer real and come in complex conjugate pairs. This justifies the picture presented in figure 1.

In what follows, we proceed to calculate ϵ_c. In terms of $V(x)$, equation (9) becomes

$$\frac{\mathrm{d}V}{\mathrm{d}x} + V^2 - \frac{\gamma^2}{4j^2} = \frac{\partial_\phi^2 h(\phi_0(x), x)}{j}. \tag{15}$$

This equation has the form of a Langevin equation with random force given by $\partial_\phi^2 h$, whose correlator is given by (see [5] for justification of $\partial_\phi^2 h$ as a random white noise)

$$\left\langle \partial_\phi^2 h(\phi_0(x), x)\partial_\phi^2 h(\phi_0(y), y)\right\rangle = \alpha\delta(x-y).$$

In accordance with the theory of Langevin equations, if

$$\frac{\mathrm{d}V}{\mathrm{d}x} + g(V) = f(x)$$

with $f(x)$ being uncorrelated at different values of x, $\langle f(x)f(x')\rangle = \alpha\delta(x-x')$, then the probability $P(v,x) = \langle\delta(v - V(x)\rangle$ of observing $V(x) = v$ at the position x obeys the Fokker–Planck equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\partial}{\partial v}\left[\frac{\alpha}{2}\frac{\partial}{\partial v} + g(v)\right]P(v,x).$$

In the present case, we are interested in the probability of observing a particular value for $V(x)$ together with the probability that a particular solution $\phi_0(x)$ of equation (5) is chosen. The relevant quantity describing this joint probability is

$$\mathcal{P}(v,x) = \left\langle \frac{\delta(v - V(x))}{\rho(x)}\right\rangle,$$

where $\rho(x)$ is the density of solutions $\phi_0(x)$ which in turn obeys the continuity equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}x} + \rho V = 0.$$

As a result, we find the Fokker–Planck equation

$$\frac{\mathrm{d}\mathcal{P}}{\mathrm{d}x} = \frac{\partial}{\partial v}\left[\frac{\alpha}{2j^2}\frac{\partial}{\partial v} + v^2 - \frac{\gamma^2}{4j^2}\right]\mathcal{P} + \lambda v\mathcal{P}.$$

Here $\lambda = 1$; however, we will keep the more general notation of λ for convenience at a later point in the calculations.

A common approach to the Fokker–Planck equations is to map them into the Schrödinger equation with the help of

$$\mathcal{P} = \exp\left[-\frac{v^3 j^2}{3\alpha} + \frac{v\gamma^2}{4\alpha}\right]\Psi.$$

This gives

$$-\frac{\mathrm{d}\Psi}{\mathrm{d}x} = \left[-\frac{\alpha}{2j^2}\frac{\partial^2}{\partial v^2} + U(v)\right]\Psi. \tag{16}$$

Here,

$$U(v) = \left[\frac{j^2}{2\alpha}v^4 - \frac{\gamma^2}{4\alpha}v^2 - v(1+\lambda) + \frac{\gamma^4}{32\alpha j^2}\right]. \tag{17}$$

The Feynman path integral formulation of the Schrödinger equation (16) allows us to find the average of V using

$$\begin{aligned}\ell\langle V\rangle &= \frac{\int \mathcal{D}V(x)\,\mathrm{e}^{-S[V]}\int_0^\ell \mathrm{d}x\,V(x)}{\int \mathcal{D}V(x)\,\mathrm{e}^{-S[V]}} \\ &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\log\int \mathcal{D}V(x)\,\mathrm{e}^{-S[V]}\Big|_{\lambda=1},\end{aligned} \tag{18}$$

for $\ell \to \infty$. Here, $S[V]$ is the imaginary time action which corresponds to the quantum mechanics, equation (16),

$$S[V] = \int_0^\ell \mathrm{d}x\left[\frac{j^2}{2\alpha}\left(\frac{\mathrm{d}V}{\mathrm{d}x}\right)^2 + U(V)\right]. \tag{19}$$

At large ℓ the path integral equation (18) is dominated by its ground-state energy. The average V may now be calculated as

$$\langle V\rangle = -\frac{\mathrm{d}E_0}{\mathrm{d}\lambda}\Big|_{\lambda=1}, \tag{20}$$

where E_0 is the quantum mechanical ground-state energy for a particle whose classical action is given by equation (19) and whose Schrödinger equation reads

$$\left[-\frac{\alpha}{2j^2}\frac{\mathrm{d}^2}{\mathrm{d}v^2} + U(v)\right]\Psi = E\Psi.$$

If $\gamma = 0$, then the ground-state energy, together with its derivative with respect to λ, can be estimated simply from dimensional analysis as $\alpha^{1/3}/j^{2/3}$, to give $\omega_c \sim \alpha^{1/3}/j^{1/6}$. This agrees with [10]. If $\gamma > 0$, then dimensional analysis is of no help, since ω_c can now depend on the dimensionless ratio $\gamma/(j\alpha)^{1/3}$. The only way to find ω_c and thus the localization length of the low-lying states is by finding the ground-state energy E_0.

By a suitable rescaling $v = y\alpha^{1/3}/j^{2/3}$, $\tilde{E} = Ej^{2/3}/\alpha^{1/3}$, we can bring the Schrödinger equation to the form

$$\left[-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \frac{1}{2}(y^2 - c^2)^2 - (1+\lambda)\,y\right]\Psi = \tilde{E}\Psi, \tag{21}$$

where

$$c = \frac{\gamma}{2\,(\alpha j)^{\frac{1}{3}}}.$$

In general, it is not possible to find the ground-state energy of this Schrödinger equation exactly. We are going to find it only in the limit when $c \gg 1$. The potential in equation (21) has two minima. The minimum at positive y is the global minimum and is located at

$$y_{\min} = c + \frac{(1+\lambda)}{4c^2} + \mathcal{O}(c^{-5}). \tag{22}$$

The simplest approximation to the ground-state energy is to set it equal to the value of the potential at the minimum. This is given by

$$\tilde{E}_{0,0} = -(1+\lambda)c - \frac{(1+\lambda)^2}{8c^2} + \mathcal{O}(c^{-5}).$$

To add quantum fluctuations to the problem, we have to consider the Schrödinger equation (21) evaluated around the minimum, equation (22). Approximating the potential of equation (21) by a quadratic potential around the point $y_{\min}$ we find the oscillator ground-state energy to be

$$\tilde{E}_{0,qf} = c + \frac{3(1+\lambda)}{8c^2} + \mathcal{O}(c^{-5}).$$

In principle, the cubic and quartic term in the expansion of the potential around the minimum also contribute to the ground-state energy, but these only have dependence upon λ at the $\mathcal{O}(c^{-5})$ level and thus do not contribute to $\langle V\rangle$ at lower orders.

Gathering the terms, it is seen that as a function of λ

$$\begin{aligned}\langle V\rangle &= -\frac{\mathrm{d}E_0}{\mathrm{d}\lambda} = -\frac{\mathrm{d}}{\mathrm{d}\lambda}(\tilde{E}_{0,0} + \tilde{E}_{0,qf})\frac{\alpha^{\frac{1}{3}}}{j^{\frac{2}{3}}} \\ &= \frac{\gamma}{2j} + \alpha\left(\lambda - \frac{1}{2}\right)\frac{1}{\gamma^2} + \mathcal{O}(\gamma^{-5}).\end{aligned}$$

Finally, substituting $\lambda = 1$, we find

$$\langle V\rangle = \frac{\gamma}{2j} + \frac{\alpha}{2\gamma^2} + \mathcal{O}(\gamma^{-5}).$$

Therefore, at large γ the energy scale ω_c is given by

$$\omega_c = \frac{\gamma}{2\sqrt{j}}. \tag{23}$$

We see that $\langle V\rangle > \gamma/2j$, and the localization length of the low-lying Hermitian states $\tilde{\psi}_n$ is consequently smaller than j/γ. This allows us to deduce that equation (6) is indeed equivalent to equation (13) at $\epsilon_n < \epsilon_c$. The localization length of ψ_n at $\epsilon_n < \epsilon_c$ can be deduced from equation (14) to be

$$\xi = \left(\langle V\rangle - \frac{\gamma}{2j}\right)^{-1} = \frac{2\gamma^2}{\alpha} + \mathcal{O}(\gamma^{-1}). \tag{24}$$

The calculation presented here justifies figure 1 at large γ. In order to see that the picture of low-lying localized states persist at all values of γ we need to show that $\langle V\rangle$ is always greater than $\gamma/2j$. This amounts to showing that in the Schrödinger equation (21) the following relation always holds true:

$$\langle y\rangle \equiv \int \mathrm{d}y|\psi_0(y)|^2 y = -\left.\frac{\mathrm{d}\tilde{E}_0}{\mathrm{d}\lambda}\right|_{\lambda=1} > c. \tag{25}$$

Here, $\psi_0(y)$ is the ground-state wavefunction of equation (21), at $\lambda = 1$. We do not know how to show this analytically. We investigated equation (21) numerically. Figure 2 shows the values of $-\mathrm{d}\tilde{E}_0/\mathrm{d}\lambda = \langle y\rangle$, at $\lambda = 1$, plotted versus c. We see that equation (25) does seem to hold.

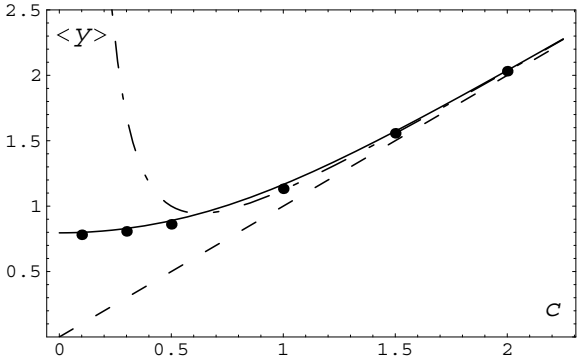

Figure 2. $\langle y\rangle$ in the ground state of equation (21) as a function of c. The dots represent $\langle y\rangle$ evaluated numerically at various values of c. The solid line is $\langle y\rangle$ calculated in the harmonic approximation to the potential in equation (21). The dashed–dotted line represents the first two terms of the perturbative calculation $c + \frac{1}{8c^2}$. Finally, the dashed line is a straight line of slope 1, which demonstrates that $\langle y\rangle > c$.

4. The Larkin length

In the analysis of equation (5) it is customary to introduce the notion of Larkin length L_{Larkin}. L_{Larkin} is defined as the size of a box in which the average $\langle\phi^2\rangle$ becomes of the order of 1. Thus at distances bigger than L_{Larkin}, $\phi_0(x)$ becomes rough, while at shorter distances it can be thought of as smooth. We are now going to see that in our problem the localization length of low-lying states of equation (6) is of the order of Larkin length.

To calculate the Larkin length, we follow the procedure described in [12]. We consider equation (5). Let us assume that ϕ is small and neglect the ϕ dependence of h. In Fourier space, this equation of motion is

$$(\mathrm{i}\gamma k + jk^2)\phi_k = -\partial_\phi h(0).$$

Thus,

$$\langle|\phi_k|^2\rangle = \frac{\langle\partial_\phi h(0)\partial_\phi h(0)\rangle}{\gamma^2k^2 + j^2k^4}.$$

Now the Larkin length is defined as the value of L_{Larkin} for which

$$1 \sim \left\langle\phi_0^2(x)\right\rangle = \int_{L_{\text{Larkin}}^{-1}}^{\infty} \frac{\mathrm{d}k}{2\pi}\frac{\alpha}{\gamma^2k^2 + j^2k^4}$$

$$\sim \frac{\alpha L_{\text{Larkin}}}{\gamma^2} - \frac{j\alpha}{\gamma^3}\tan^{-1}(\gamma L_{\text{Larkin}}/j).$$

From the asymptotic form of $\tan^{-1}$ in the large-γ limit it is found that

$$L_{\text{Larkin}} \sim \frac{\gamma^2}{\alpha} + \frac{\pi}{2}j\gamma^{-1} - j^2\alpha\gamma^{-4} + \mathcal{O}(\gamma^{-7}).$$

At leading order L_{Larkin} turns out to be proportional to the localization length obtained above, equation (24). This coincides with the behaviour of the pinned charge density waves without the background current [10]. However, the localization length and the Larkin length have different functional dependence on γ in lower orders.

5. Density of states

In this section, we calculate the density of low-lying states for our problem. Due to the form of the potential, equation (17), which does not contain a cubic term, the density of low-lying states maybe easily calculated using the methods of [5]. This is not immediately obvious, so in this section we will repeat the argument.

At energies below an energy of the order of $\langle V\rangle$ the energy eigenvalues are real. Above, the eigenvalues trace out a one-dimensional spectrum in the complex plane (figure 1). Let us concentrate on low-lying states $\epsilon_n < \epsilon_c$, which are all real. To calculate their density of states $\rho(\epsilon)$ we use the equivalence of equations (6) and (13) in this regime.

The integrated density of low-lying states can be found using the discussion in [5] as

$$N(\omega) = \left\langle \delta\left(\int_{x_1}^{x_2} \mathrm{d}x\ V(x) + a\right)\right\rangle, \qquad a = -\log(\omega),$$

with the average being over realizations of $V(x)$. The density of states is then obtained from $\rho(\omega) = \mathrm{d}N/\mathrm{d}\omega$.

Through a standard representation of the δ-function

$$N(\omega) = \left\langle \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{2\pi} \exp\left(\mathrm{i}\alpha\left(\int_{x_1}^{x_2} \mathrm{d}x\ V(x) + a\right)\right)\right\rangle,$$

and defining $\Lambda(\alpha)$ by

$$\exp(-\ell\Lambda(\alpha)) = \left\langle \exp\left(\mathrm{i}\alpha \int_{x_1}^{x_2} \mathrm{d}x\ V(x)\right)\right\rangle, \tag{26}$$

with $\ell = x_2 - x_1$ the integrated density of states is

$$N(\omega) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\alpha}{2\pi} \exp(\mathrm{i}\alpha a - \ell\Lambda(\alpha)).$$

At large ℓ, $\Lambda(\alpha)$ is independent of ℓ as will be argued below. We assume in the following that a certain value of ℓ maximizes the probability of observing a fluctuation of size $-\log\omega$. This is justified, since due to the form of the potential $U(V)$, equation (17), it is not very likely to observe a very large negative fluctuation, and it is more likely to see a smaller negative fluctuation, which exists over a longer interval of x. We now approximate $N(\omega)$ by its value at the saddle point, determined from

$$\mathrm{i}a - \ell\frac{\partial\Lambda}{\partial\alpha} = 0,$$

where $\alpha = \alpha_0(\ell)$ at the saddle point. Thus, $N(\omega) \sim \exp(\mathrm{i}a\alpha_0 - \ell\Lambda(\alpha_0(\ell)))$ and the maximization with respect to ℓ results in

$$\begin{aligned} 0 &= \mathrm{i}a\frac{\partial\alpha_0}{\partial\ell} - \Lambda(\alpha_0(\ell)) - \ell\frac{\partial\Lambda}{\partial\alpha}\frac{\partial\alpha_0}{\partial\ell} \\ &= -\Lambda(\alpha_0(\ell)). \end{aligned} \tag{27}$$

Defining $\beta_0 \equiv -\mathrm{i}\alpha_0$, it is seen that $N(\omega) \sim \omega^{\beta_0}$ where β_0 should be chosen such that $\Lambda(\beta_0)$ vanishes.

Making use of our results from section 3 where the Schrödinger equation (16) was found with the potential taking the form equation (17), equation (26) becomes

$$\exp(-\ell\Lambda(\beta_0)) = \frac{\int \mathcal{D}V(x)\exp(-S[V])\exp\left(-\beta_0\int_{x_1}^{x_2}\mathrm{d}x\ V\right)}{\int \mathcal{D}V(x)\exp(-S[V])}, \tag{28}$$

with $S[V]$ given by equation (19). At large lengths ℓ, the path integrals are dominated by the ground-state energy, i.e.,

$$\exp(-\ell\Lambda(\beta_0)) = \frac{\exp(-\ell E_0\{-1-\lambda+\beta_0\})}{\exp(-\ell E_0\{-1-\lambda\})},$$

where the notation $E_0\{s\}$ represents the ground state of the Schrödinger equation (16) with the potential equation (17) where s is substituted in place of $1+\lambda$. Furthermore, since all terms except the linear are invariant under $V(x) \to -V(x)$, we see that $E_0\{s\} = E_0\{-s\}$. Thus, $\beta_0 = 2+2\lambda$ obviously solves equation (27). Note that the above argument hinged on the fact that the potential $U(V)$ does not contain a cubic term. We conclude that with $\lambda = 1$

$$N(\omega) \sim \omega^4 = \epsilon^2,$$

and

$$\rho(\omega) \sim \omega^3, \qquad \rho(\epsilon) = \frac{\mathrm{d}N(\epsilon)}{\mathrm{d}\epsilon} \sim \epsilon. \tag{29}$$

We would like to compare this with the result obtained for the pinned charge density wave problem with $\gamma = 0$. There $\rho(\omega) \sim \omega^4$ if ϕ_0 is a global minimum of the energy functional, equation (1), and $\rho(\omega) \sim \omega^3$ if the minimum is local. At $\gamma > 0$, however, equation (5) is no longer a minimization condition of any functional and the notion of global minimum no longer exists.

6. Driven pinned charge density waves with background current

Consider a pinned charge density wave driven by an electric field at frequency ω_0 (see, e.g., [18] for a review),

$$\nu\frac{\mathrm{d}\phi}{\mathrm{d}t} - j\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - \gamma\frac{\mathrm{d}\phi}{\mathrm{d}x} + \partial_\phi h(\phi, x) = E(x)\cos(\omega_0 t).$$

To find the linear response to a small electric field, we write $\phi = \phi_0(x) + \psi(x,t)$, where $\phi_0(x)$ satisfies equation (5), and find

$$\left[\nu\frac{\mathrm{d}}{\mathrm{d}t} - j\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \gamma\frac{\mathrm{d}}{\mathrm{d}x} + \partial_\phi^2 h(\phi_0(x), x)\right]\psi(x,t) = E(x)\,\mathrm{e}^{i\omega_0 t}.$$

The solution to this equation can now easily be found

$$\phi(x,t) = \phi_0(x) + \mathrm{e}^{\mathrm{i}\omega_0 t}\sum_n \int \mathrm{d}y\; E(y)\frac{\psi_n(x)\psi_n(y)}{\mathrm{i}\nu\omega_0 + \epsilon_n},$$

where ψ_n and ϵ_n are defined in equation (6). The current carried by the charge density wave is $I \sim \frac{\partial\psi}{\partial t}$, and is thus given by

$$I(x) \sim \mathrm{i}\omega_0\,\mathrm{e}^{\mathrm{i}\omega_0 t}\sum_n \int \mathrm{d}y\; E(y)\frac{\psi_n(x)\psi_n(y)}{\mathrm{i}\nu\omega_0 + \epsilon_n}.$$

The properties of ψ_n, ϵ_n found in this paper help to determine the response of the charge density wave to a small external electric field. For example, if the electric field $E(x)$ is uniform in space, we can use the results of this paper which show that at $\epsilon_n < \epsilon_c$, the wavefunctions $\psi_n(x)$ are all localized with the same localization length ξ. Therefore, $\int \mathrm{d}x\; \psi_n(x) \propto \sqrt{\xi}$ and is independent of n, as long as $\epsilon_n < \epsilon_c$. On the other hand, the wavefunctions which

correspond to large $\mathrm{Re}\,\epsilon_n$ are delocalized and oscillate fast so that $\int \mathrm{d}x\ \psi_n(x)$ goes to zero with increasing n quickly. Therefore, the current reduces to

$$I \sim \mathrm{i}\omega_0\, \mathrm{e}^{\mathrm{i}\omega_0 t} \int_0^{\epsilon_c} \mathrm{d}\epsilon \rho(\epsilon) \frac{1}{\mathrm{i}\nu\omega_0 + \epsilon}$$
$$\sim \mathrm{i}\omega_0\, \mathrm{e}^{\mathrm{i}\omega_0 t} \left[\epsilon_c - \mathrm{i}\nu\omega_0 \log \frac{\epsilon_c + \mathrm{i}\nu\omega_0}{\mathrm{i}\nu\omega_0} \right], \tag{30}$$

where $\rho(\epsilon) \sim \epsilon$, as found in this paper, equation (29).

7. Conclusions

We have demonstrated how the problem of normal mode oscillations of the one-dimensional pinned charge density waves with background current may be solved by mapping into chiral random Hamiltonians. Using the methods of Comtet *et al* [17] we have determined the localization length ξ of the low lying normal modes. This localization length turns out to be proportional to the Larkin length of this system. The density of the lowest lying states has also been obtained, giving the power law $\rho(\omega) \sim \omega^3$. This result differs from the result $\rho(\omega) \sim \omega^4$ of [8, 10] due to the fact that turning on background current removes the notion of a global ground state.

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References

[1] P W Anderson 1958 *Phys. Rev.* **109** 1492
[2] Efetov K B 1997 *Supersymmetry in Disorder and Chaos* (Cambridge: Cambridge University Press)
[3] Altland A and Zirnbauer M R 1997 *Phys. Rev.* B **55** 1142
[4] Zirnbauer M R 1996 *J. Math. Phys.* **37** 4986
[5] Gurarie V and Chalker J T 2003 *Phys. Rev.* B **68** 134207
[6] Gruner G 1994 *Density Waves in Solids* (Reading, MA: Addison-Wesley Longman)
[7] Fukuyama H and Lee P A 1978 *Phys. Rev.* B **17** 535
[8] Aleiner I L and Ruzin I M 1994 *Phys. Rev. Lett.* **72** 1056
[9] Fogler M 2002 *Phys. Rev. Lett.* **88** 186402
[10] Gurarie V and Chalker J T 2002 *Phys. Rev. Lett.* **89** 136801
[11] Titov M, Brouwer P W, Furusaki A and Mudry C 2001 *Phys. Rev.* B **63** 235318
[12] Radzihovsky L and Toner J 1998 *Phys. Rev. Lett.* **81** 3711
[13] Balents L and Fisher M P A 1995 *Phys. Rev. Lett.* **75** 4270
[14] Giamarchi T and Le Doussal P 1996 *Phys. Rev. Lett.* **76** 3408
[15] Ya Goldsheid I and Khoruzhenko B A 1998 *Phys. Rev. Lett.* **80** 2897
[16] Hatano N and Nelson D R 1996 *Phys. Rev. Lett.* **77** 570
[17] Comtet A, Debois J and Monthus C 1995 *Ann. Phys.* **239** 312
[18] Brazovskii S and Nattermann T 2004 *Adv. Phys.* **53** 177